

Perturbations of continuous-time Markov chains

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Abstract. The equivalence of regularity of a Q -matrix with its bounded perturbations is proved and an integration by parts formula is established for the associated Feller minimal transition functions.

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1 Introduction

One of the basic questions in studying continuous-time Markov chains is to find the regularity criterion, i.e., to investigate the conditions under which the given Q -matrix is regular, or, equivalently, the corresponding Feller minimal process is honest in the sense that the corresponding transition function $P(t) = \{P_{ij}(t); i, j \in \mathbb{N}\}$ satisfies $\sum_{j=0}^{\infty} P_{ij}(t) = 1$ for all $i \geq 0$ and $t \geq 0$. Here we assume the chain has state space $\mathbb{N} := \{0, 1, 2, \dots\}$. We refer to Anderson (1991) and Chen (2004) for the general theory of continuous-time Markov chains. In this note we show that the regularity property is preserved under a bounded perturbation of the Q -matrix. We also establish an integration by parts formula for the corresponding Feller minimal processes without the regularity condition.

Given two Q -matrices $R = (r_{ij}; i, j \in \mathbb{N})$ and $A = (a_{ij}; i, j \in \mathbb{N})$, we call $Q = (q_{ij}; i, j \in \mathbb{N}) := R + A$ the *perturbation of R by A* . Throughout this note, we assume all Q -matrices are stable and conservative.

The main purpose of this note is to prove the following theorems:

Theorem 1.1 *Suppose that A is a bounded Q -matrix. Then $Q = R + A$ is regular if and only if R is regular.*

Theorem 1.2 *Let $Q(t) = \{Q_{ij}(t); i, j \in \mathbb{N}\}$ and $R(t) = \{R_{ij}(t); i, j \in \mathbb{N}\}$ be the Feller minimal transition functions of Q and R , respectively. Then we have the following integration by parts formula*

$$\sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k Q_{kj}(t-s) ds = \sum_{l \in \mathbb{N}, l \neq i} \int_0^t R_{il}(t-v) a_{lm} Q_{mj}(v) dv + R_{ij}(t) - Q_{ij}(t). \quad (1.1)$$

In particular, when $\sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k Q_{kj}(t-s) ds < \infty$, we can rewrite (1.1) as

$$Q(t) - R(t) = \int_0^t R(s) A Q(t-s) ds. \quad (1.2)$$

The perturbation theory of infinitesimal generators has been a very useful tool in the hands of analysts and physicists. A considerable amount of research has been done on the perturbation of linear operators on a Banach space. The effect on a semigroup by adding a linear operator to its infinitesimal generator was studied by Phillips (1952) and Yan (1988). However, these authors did not show the equivalence of the regularity of a Q -matrix with its bounded perturbations. The integration by parts formula (1.1) was given by Chen (2004, p510) under a stronger condition. The Q -matrix of the branching processes with immigration and/or resurrection introduced in Li and Chen (2006) can be regarded as the perturbations of a given branching Q -matrix.

Example 1.3 Let $R = (r_{ij}; i, j \in \mathbb{N})$ be a branching Q -matrix given by

$$r_{ij} = \begin{cases} ib_{j-i+1} & j \geq i-1, i \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$b_j \geq 0 \ (j \neq 0), \quad \sum_{n \neq 1}^{\infty} b_j = -b_1 \geq 0.$$

Let $A = (a_{ij}; i, j \in \mathbb{N})$ be a bounded Q -matrix given by

$$a_{ij} = \begin{cases} c_{j-i+1} & j \geq i, i \geq 1; \\ h_j & j \geq 0, i = 0; \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{cases} h_j \geq 0 \ (j \neq 0), & \sum_{j=1}^{\infty} h_j = -h_0 \geq 0; \\ c_j \geq 0 \ (j \neq 0), & \sum_{j=1}^{\infty} c_j = -c_0 \geq 0. \end{cases}$$

Then the Q -matrix $Q = (q_{ij}; i, j \in \mathbb{Z}^+) := R + A$ is called a branching Q -matrix with immigration and resurrection. The corresponding continuous-time Markov chain is called a branching process with immigration and resurrection. Note that the regularity criterion of the branching Q -matrix R is given by Harris (1963). Since A is a bounded Q -matrix, by Theorem 1.1 we see Q is regular if and only if R is regular. This simplifies considerably the proof of Theorem 2.1 in Li and Chen (2006).

2 Bounded perturbations

In this section, we assume A is a bounded Q -matrix. We shall prove that the regularity of R and Q are equivalent. Let $\gamma = \sup_i a_i = -\inf_i a_{ii}$. Let $q'_{ii} = \gamma - a_i > 0$ and $q'_{ij} = q_{ij}$ for $i \neq j$. Let $a'_{ij} = a_{ij} + \gamma \delta_{ij} > 0$. Then we have $q'_{ik} = a'_{ik} + (1 - \delta_{ik}) r_{ik} > 0$,

Proposition 2.1 *The backward Kolmogorov equation of Q is equivalent to the following equation:*

$$Q_{ij}(t) = \sum_{k \in \mathbb{N}} \int_0^t e^{-(r_i + \gamma)(t-s)} q'_{ik} Q_{kj}(s) ds + \delta_{ij} e^{-(r_i + \gamma)t}. \quad (2.3)$$

Proof. Suppose that $Q(t) = \{Q_{ij}(t); i, j \in \mathbb{N}\}$ is a solution of the backward Kolmogorov equation $\partial_t Q(t) = QQ(t)$. Then

$$\partial_t Q_{ij}(t) + (r_i + \gamma)Q_{ij}(t) = \sum_{k \in \mathbb{N}} q'_{ik} Q_{kj}(t).$$

Multiplying both sides by the integrating factor $e^{(r_i + \gamma)t}$, we find

$$\partial_t (e^{(r_i + \gamma)t} Q_{ij}(t)) = e^{(r_i + \gamma)t} \sum_{k \in \mathbb{N}} q'_{ik} Q_{kj}(t).$$

Integrating and dividing both sides by $e^{(r_i + \gamma)t}$ give (2.3). Conversely, suppose $Q_{ij}(t)$ is a solution of (2.3). By differentiating both sides of the equation we get the backward Kolmogorov equation $\partial_t Q(t) = QQ(t)$. \square

Let $Q(t) = \{Q_{ij}(t); i, j \in \mathbb{N}\}$ and $R(t) = \{R_{ij}(t); i, j \in \mathbb{N}\}$ be the minimal transition functions of Q and R , respectively. By the *second successive approximation scheme*; see, e.g., Chen (2004, p64), we see

$$Q_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t) \quad \text{and} \quad R_{ij}(t) = \sum_{n=0}^{\infty} R_{ij}^{(n)}(t), \quad (2.4)$$

where

$$R_{ij}^{(0)}(t) = \delta_{ij} e^{-r_i t}, \quad R_{ij}^{(n+1)}(t) = \sum_{k \neq i} \int_0^t e^{-r_i(t-s)} r_{ik} R_{kj}^{(n)}(s) ds \quad (2.5)$$

and

$$Q_{ij}^{(0)}(t) = \delta_{ij} e^{-(r_i + \gamma)t}, \quad Q_{ij}^{(n+1)}(t) = \sum_{k \in \mathbb{N}} \int_0^t e^{-(r_i + \gamma)(t-s)} q'_{ik} Q_{kj}^{(n)}(s) ds. \quad (2.6)$$

Lemma 2.2 *For any $n \geq 0$ we have*

$$Q_{ij}^{(n)}(t) = \sum_{p=0}^{n-1} \sum_{l, k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{il}^{(n-p-1)}(t-s) a'_{lk} Q_{kj}^{(p)}(s) ds + R_{ij}^{(n)}(t) e^{-\gamma t} \quad (2.7)$$

with $\sum_{p=0}^{-1} = 0$ by convention.

Proof. For $n = 0$, we have (2.7) trivially. Suppose that (2.7) holds for $n = 0, 1, \dots, m$. Recall that $q'_{ik} = a'_{ik} + (1 - \delta_{ik})r_{ik}$. By the second equality in (2.6) we have

$$\begin{aligned} Q_{ij}^{(m+1)}(t) &= \sum_{k \neq i} \int_0^t e^{-(r_i + \gamma)(t-s)} r_{ik} Q_{kj}^{(m)}(s) ds \\ &\quad + \sum_{k \in \mathbb{N}} \int_0^t e^{-(r_i + \gamma)(t-s)} a'_{ik} Q_{kj}^{(m)}(s) ds \\ &=: I_1 + I_2. \end{aligned}$$

By (2.7) and (2.5) we have

$$\begin{aligned} I_1 &= \sum_{k \neq i} \int_0^t e^{-(r_i + \gamma)(t-s)} r_{ik} \left[\sum_{p=0}^{m-1} \sum_{l, r \in \mathbb{N}} \int_0^s e^{-\gamma(s-u)} R_{kl}^{(m-p-1)}(s-u) a'_{lr} Q_{rj}^{(p)}(u) du \right] ds \\ &\quad + \sum_{k \neq i} \int_0^t e^{-(r_i + \gamma)(t-s)} r_{ik} R_{kj}^{(m)}(s) e^{-\gamma s} ds \\ &= \sum_{p=0}^{m-1} \sum_{l, r \in \mathbb{N}} \int_0^t e^{-\gamma(t-u)} \left[\sum_{k \neq i} \int_u^t e^{-r_i(t-s)} r_{ik} R_{kl}^{(m-p-1)}(s-u) ds \right] a'_{lr} Q_{rj}^{(p)}(u) du \\ &\quad + \sum_{k \neq i} \int_0^t e^{-(r_i + \gamma)(t-s)} r_{ik} R_{kj}^{(m)}(s) e^{-\gamma s} ds \\ &= \sum_{p=0}^{m-1} \sum_{l, r \in \mathbb{N}} \int_0^t e^{-\gamma(t-u)} R_{il}^{(m-p)}(t-u) a'_{lr} Q_{rj}^{(p)}(u) du + R_{ij}^{(m+1)}(t) e^{-\gamma t}. \end{aligned}$$

On the other hand, using the first equality in (2.5) we obtain

$$\begin{aligned} I_2 &= \sum_{k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{ii}^{(0)}(t-s) a'_{ik} Q_{kj}^{(m)}(s) ds \\ &= \sum_{l, k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{il}^{(0)}(t-s) a'_{lk} Q_{kj}^{(m)}(s) ds. \end{aligned}$$

Summing up the above expressions of I_1 and I_2 , we see (2.7) also holds when $n = m + 1$. That gives the desired result. \square

Proposition 2.3 Let $Q(t) = \{Q_{ij}(t); i, j \in \mathbb{N}\}$ and $R(t) = \{R_{ij}(t); i, j \in \mathbb{N}\}$ be the minimal transition functions of Q and R , respectively. Then $Q_{ij}(t)$ is the unique solution of the following equation

$$Q_{ij}(t) = \sum_{l, k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{il}(t-s) a'_{lk} Q_{kj}(s) ds + R_{ij}(t) e^{-\gamma t}. \quad (2.8)$$

Proof. We first prove the uniqueness of (2.8). Let $\tilde{Q}_{ij}(t)$ be another solution of (2.8). Let $c_{ij}(t) = |Q_{ij}(t) - \tilde{Q}_{ij}(t)|$ and $c_j(t) = \sup_i c_{ij}(t)$. Then we have

$$c_{ij}(t) \leq \sum_{l, k \in \mathbb{N}} \int_0^t R_{il}(t-s) a'_{lk} c_{kj}(s) ds.$$

Taking the supremum we have

$$c_j(t) \leq \sup_i \sum_{l,k \in \mathbb{N}} \int_0^t R_{il}(t-s) a'_{lk} c_j(s) ds = \gamma \int_0^t c_j(s) ds.$$

Using Gronwall's inequality we have that $c_j(t) = 0$. Thus (2.8) has at most one solution.

Next we will show that $Q_{ij}(t)$ satisfies (2.8). Using (2.4) and (2.7) we have

$$Q_{ij}(t) = \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{l,k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{il}^{(n-p)}(t-s) a'_{lk} Q_{kj}^{(p)}(s) ds + \sum_{n=0}^{\infty} R_{ij}^{(n)}(t) e^{-\gamma t}.$$

Interchanging the order of summation and using (2.4) again we obtain

$$\begin{aligned} Q_{ij}(t) &= \sum_{l,k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} \sum_{n=p}^{\infty} R_{il}^{(n-p)}(t-s) a'_{lk} \sum_{p=0}^{\infty} Q_{kj}^{(p)}(s) ds + R_{ij}(t) e^{-\gamma t} \\ &= \sum_{l,k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{il}(t-s) a'_{lk} Q_{kj}(s) ds + R_{ij}(t) e^{-\gamma t}. \end{aligned}$$

That completes the proof. \square

Proof of Theorem 1.1. Summing up both sides of (2.8) over j , we see that $x_i(t) := \sum_{j=0}^{\infty} Q_{ij}(t)$ is a solution to the following equation:

$$x_i(t) = \sum_{l,k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{il}(t-s) a'_{lk} x_k(s) ds + e^{-\gamma t} \sum_{j=0}^{\infty} R_{ij}(t). \quad (2.9)$$

Suppose that R is regular. Then we have $\sum_{j=0}^{\infty} R_{ij}(t) = 1$, so $x_i(t) \equiv 1$ is a solution of (2.9). Let $\tilde{x}_i(t)$ be another solution of (2.9). Set $c_i(t) = |x_i(t) - \tilde{x}_i(t)|$ and $c(t) = \sup_i c_i(t)$. By (2.9) we obtain

$$c_i(t) \leq \sum_{l,k \in \mathbb{N}} \int_0^t R_{il}(t-s) a'_{lk} c_k(s) ds.$$

Taking the supremum we get

$$c(t) \leq \sup_i \sum_{l,k \in \mathbb{N}} \int_0^t R_{il}(t-s) a'_{lk} c(s) ds = \gamma \int_0^t c(s) ds.$$

Using Gronwall's inequality we have $c(t) = 0$. Then we see $x_i(t) \equiv 1$ is the unique solution to (2.9). Hence Q is regular.

Conversely, suppose that Q is regular. Then $x_i(t) = \sum_{j=0}^{\infty} Q_{ij}(t) = 1$. Let $y_i(t) = \sum_{j=0}^{\infty} R_{ij}(t)$. From (2.9) we have

$$1 - e^{-\gamma t} \leq \int_0^t \gamma e^{-\gamma(t-s)} y_i(t-s) ds.$$

Then we must have $y_i(t) \equiv 1$, so R is regular. \square

3 Integration by parts formula

Recall that $R(t)$ and $Q(t)$ are the Feller minimal transition functions of R and Q , respectively. By the second successive approximation scheme; see, e.g. Chen (2004, p64) we have

$$Q_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t), \quad (3.10)$$

where

$$Q_{ij}^{(0)}(t) = \delta_{ij}e^{-q_i t}, \quad Q_{ij}^{(n+1)}(t) = \sum_{k \neq i} \int_0^t e^{-q_i(t-s)} q_{ik} Q_{kj}^{(n)}(s) ds. \quad (3.11)$$

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, P_i)$ be a realization of $(R_{ij}(t))_{t \geq 0}$.

Lemma 3.1 *Let σ_s^t denote the number of jumps of the trajectory $t \mapsto \xi_t$ on the interval $(s, t]$. Then for $n \geq 0$ we have*

$$\begin{aligned} Q_{ij}^{(n)}(t) = & \sum_{p=0}^{n-1} \sum_{k \in \mathbb{N}, l \neq k} \int_0^t P_i(M_{t-s}^0; A_{n-p-1,k}(0, t-s)) a_{kl} Q_{lj}^{(p)}(s) ds \\ & + P_i(M_t^0; A_{n,j}(0, t)) \end{aligned} \quad (3.12)$$

with $\sum_{p=0}^{-1} = 0$ by convention, where $A_{n,j}(s, t) = \{\sigma_s^t = n, \xi_t = j\}$ and $M_t^r = e^{-\int_r^t a(\xi_s) ds}$.

Proof. For $n = 0$ we have (3.12) trivially. Suppose that (3.12) holds for $n = 0, 1, \dots, m$. By (3.11) we have

$$\begin{aligned} Q_{ij}^{(m+1)}(t) = & \sum_{k \neq i} \int_0^t e^{-q_i(t-s)} r_{ik} Q_{kj}^{(m)}(s) ds \\ & + \sum_{k \neq i} \int_0^t e^{-q_i(t-s)} a_{ik} Q_{kj}^{(m)}(s) ds =: I_1 + I_2. \end{aligned}$$

Denote $\tau = \inf\{t \geq 0 : \xi_t \neq \xi_0\}$. By the Markov property we have

$$\begin{aligned} I_1 = & \sum_{k \neq i} \int_0^t e^{-q_i(t-s)} r_{ik} \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_0^s P_k(M_{s-v}^0; A_{m-1-p,r}(0, s-v)) a_{rl} Q_{lj}^{(p)}(v) dv ds \\ & + \sum_{k \neq i} \int_0^t e^{-q_i(t-s)} r_{ik} P_k(M_s^0; A_{m,j}(0, s)) ds \\ = & \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_0^t \sum_{k \neq i} \int_v^t e^{-q_i(t-s)} r_{ik} P_k(M_{s-v}^0; A_{m-1-p,r}(0, s-v)) a_{rl} Q_{lj}^{(p)}(v) ds dv \\ & + \sum_{k \neq i} \int_0^t e^{-a_i(t-s)} r_i^{-1} r_{ik} P_k(M_s^0; A_{m,j}(0, s)) r_i e^{-r_i(t-s)} ds \end{aligned}$$

$$\begin{aligned}
&= \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_0^t \left[\sum_{k \neq i} \int_0^{t-v} e^{-a_i s} r_i^{-1} r_{ik} P_k(M_{t-v-s}^0; A_{m-1-p,r}(0, t-v-s)) \right. \\
&\quad \left. r_i e^{-r_i s} ds \right] a_{rl} Q_{lj}^{(p)}(v) dv + \sum_{k \neq i} \int_0^t e^{-a_i s} r_i^{-1} r_{ik} P_k(M_{t-s}^0; A_{m,j}(0, t-s)) r_i e^{-r_i s} ds \\
&= \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_0^t P_i \left[e^{-a_i \tau} P_{\xi_\tau}(M_{t-v-\tau}^0; A_{m-1-p,r}(0, t-v-\tau)) \right] a_{rl} Q_{lj}^{(p)}(v) dv \\
&\quad + P_i \left[e^{-a_i \tau} P_{\xi_\tau}(M_{t-\tau}^0; A_{m,j}(0, t-\tau)) \right] \\
&= \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_0^t P_i \left[e^{-a_i \tau} P_i(M_{t-v}^\tau 1_{A_{m-p,r}(\tau, t-v)} | \mathcal{F}_\tau) \right] a_{rl} Q_{lj}^{(p)}(v) dv \\
&\quad + P_i \left[e^{-a_i \tau} P_i(M_t^\tau 1_{A_{m+1,j}(\tau, t)} | \mathcal{F}_\tau) \right] \\
&= \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_0^t P_i(M_{t-v}^0; A_{m-p,r}(0, t-v)) a_{rl} Q_{lj}^{(p)}(v) dv \\
&\quad + P_i(M_t^0; A_{m+1,j}(0, t)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
I_2 &= \sum_{l \neq i} \int_0^t e^{-q_i(t-v)} a_{il} Q_{lj}^{(m)}(v) dv \\
&= \sum_{l \neq i} \int_0^t P_i \left(e^{-a_i(t-v)} 1_{\{\sigma_0^{t-v}=0\}} \right) a_{il} Q_{lj}^{(m)}(v) dv \\
&= \sum_{r \in \mathbb{N}, l \neq r} \int_0^t P_i(M_{t-v}^0; A_{0,r}(0, t-v)) a_{rl} Q_{lj}^{(m)}(v) dv. \tag{3.13}
\end{aligned}$$

Summing up the above expressions of I_1 and I_2 we see (3.12) also holds when $n = m + 1$. That gives the desired result. \square

Theorem 3.2 *The Feller minimal transition functions $Q(t)$ and $R(t)$ satisfy the following equation*

$$Q_{ij}(t) = \sum_{k \in \mathbb{N}, l \neq k} \int_0^t P_i(M_{t-s}^0 1_{\{\xi_{t-s}=k\}}) a_{kl} Q_{lj}(s) ds + P_i(M_t^0 1_{\{\xi_t=j\}}). \tag{3.14}$$

Proof. Using (3.10) and (3.12) we have

$$Q_{ij}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \sum_{k \in \mathbb{N}, l \neq k} \int_0^t P_i(M_{t-s}^0; A_{k,n-m-1}(0, t-s)) a_{kl} Q_{lj}^{(m)}(s) ds + P_i(M_t^0 1_{\{\xi_t=j\}}).$$

Interchanging the order of summation we see (3.14) holds. \square

Proof of Theorem 1.2. By the Markov property of $\{\xi_t : t \geq 0\}$,

$$\begin{aligned}
& \sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k P_k(M_{t-s}^0 1_{\{\xi_{t-s}=j\}}) ds \\
&= \int_0^t P_i \left[a(\xi_s) P_{\xi_s}(M_{t-s}^0 1_{\{\xi_{t-s}=j\}}) \right] ds \\
&= \int_0^t P_i \left[a(\xi_s) P_i(M_t^s 1_{\{\xi_t=j\}} | \mathcal{F}_s) \right] ds \\
&= \int_0^t P_i \left[a(\xi_s) M_t^s 1_{\{\xi_t=j\}} \right] ds \\
&= P_i \left[1_{\{\xi_t=j\}} \int_0^t a(\xi_s) e^{-\int_s^t a(\xi_u) du} ds \right] \\
&= P_i \left[1_{\{\xi_t=j\}} \left(1 - e^{-\int_0^t a(\xi_u) du} \right) \right] \\
&= R_{ij}(t) - P_i(M_t^0 1_{\{\xi_t=j\}}).
\end{aligned}$$

On the other hand, by the Markov property, we have

$$\begin{aligned}
& \sum_{l \in \mathbb{N}, m \neq l} \int_0^t R_{il}(t-v) a_{lm} Q_{mj}(v) dv \\
&= \sum_{l \in \mathbb{N}, m \neq l} P_i \left[\int_0^t 1_{\{\xi_{t-v}=l\}} a_{lm} Q_{mj}(v) \left(1 - e^{-\int_0^{t-v} a(\xi_u) du} \right) dv \right] \\
&\quad + \sum_{l \in \mathbb{N}, m \neq l} P_i(M_{t-v}^0 1_{\{\xi_{t-v}=l\}}) a_{lm} Q_{mj}(v) dv \\
&= \sum_{l \in \mathbb{N}, m \neq l} P_i \left[\int_0^t 1_{\{\xi_{t-v}=l\}} a_{lm} Q_{mj}(v) dv \int_0^{t-v} a(\xi_s) e^{-\int_s^{t-v} a(\xi_u) du} ds \right] \\
&\quad + \sum_{l \in \mathbb{N}, m \neq l} P_i(M_{t-v}^0 1_{\{\xi_{t-v}=l\}}) a_{lm} Q_{mj}(v) dv \\
&= \sum_{l \in \mathbb{N}, m \neq l} P_i \left[\int_0^t a(\xi_s) \int_0^{t-s} M_{t-v}^s 1_{\{\xi_{t-v}=l\}} a_{lm} Q_{mj}(v) dv ds \right] \\
&\quad + \sum_{l \in \mathbb{N}, m \neq l} P_i(M_{t-v}^0 1_{\{\xi_{t-v}=l\}}) a_{lm} Q_{mj}(v) dv \\
&= \sum_{l \in \mathbb{N}, m \neq l} P_i \left\{ \int_0^t a(\xi_s) P_i \left[\int_0^{t-s} M_{t-v}^s 1_{\{\xi_{t-v}=l\}} a_{lm} Q_{mj}(v) dv | \mathcal{F}_s \right] ds \right\} \\
&\quad + \sum_{l \in \mathbb{N}, m \neq l} P_i(M_{t-v}^0 1_{\{\xi_{t-v}=l\}}) a_{lm} Q_{mj}(v) dv \\
&= P_i \left\{ \int_0^t a(\xi_s) \sum_{l \in \mathbb{N}, m \neq l} P_{\xi_s} \left[\int_0^{t-s} M_{t-s-v}^0 1_{\{\xi_{t-s-v}=l\}} a_{lm} Q_{mj}(v) dv \right] ds \right\} \\
&\quad + \sum_{l \in \mathbb{N}, m \neq l} P_i(M_{t-v}^0 1_{\{\xi_{t-v}=l\}}) a_{lm} Q_{mj}(v) dv \\
&= \sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k \sum_{l \in \mathbb{N}, m \neq l} \int_0^{t-s} P_k(M_{t-s-v}^0 1_{\{\xi_{t-s-v}=l\}}) a_{lm} Q_{mj}(v) dv ds
\end{aligned}$$

$$+ \sum_{l \in \mathbb{N}, m \neq l} P_i \left(M_{t-v}^0 1_{\{\xi_{t-v}=l\}} \right) a_{lm} Q_{mj}(v) dv.$$

By the above two equations and (3.14) we obtain (1.1). Suppose that

$$\sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k Q_{kj}(t-s) ds < \infty.$$

Then subtracting it from both sides of (1.1) yields (1.2). \square

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